

SOME NEW GENERALIZED RESULTS ON OSTROWSKI TYPE INTEGRAL INEQUALITIES WITH APPLICATION

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ABSTRACT. The aim of this paper is to establish some new inequalities similar to the Ostrowski's inequalities which are more generalized than the inequalities of Dragomir and Cerone. The current article obtains bounds for the deviation of a function from a combination of integral means over the end intervals covering the entire interval. Some new perturbed results are obtained. Application for cumulative distribution function is also discussed.

1. INTRODUCTION

In 1938, Ostrowski [13] established an interesting integral inequality associated with differentiable mappings. This Ostrowski inequality has powerful applications in numerical integration, probability and optimization theory, stochastic, statistics, information and integral operator theory. A number of Ostrowski type inequalities have been derived by Cerone [1], [2] and Cheng [3] with applications in Numerical analysis and Probability. Dragomir et.al [5] combined Ostrowski and Grüss inequality to give a new inequality which they named Ostrowski-Grüss type inequality. Milovanović and Pecarić [12] gave the first generalization of Ostrowski's inequality. More recent results concerning the generalizations of Ostrowski inequality are given by Liu [11], Hussain [10] and Qayyum [16]. In this paper, we will extend and generalize the results of Cerone [1] and Dragomir et.al [5]-[8] by using a new kernel.

Let $S(f; a, b)$ be defined by

$$S(f; a, b) := f(x) - M(f; a, b), \quad (1.1)$$

where

$$M(f; a, b) = \frac{1}{b-a} \int_a^b f(x) dx \quad (1.2)$$

is the integral mean of f over $[a, b]$. The functional $S(f; a, b)$ represents the deviation of $f(x)$ from its integral mean over $[a, b]$.

Ostrowski [13] proved the following integral inequality:

Theorem 1. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) , whose derivative $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e. $\|f'\|_\infty =$

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$\sup_{t \in [a, b]} |f'(t)| < \infty$, then

$$|S(f; a, b)| \leq \left[\left(\frac{b-a}{2} \right)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \frac{M}{b-a} \quad (1.3)$$

for all $x \in [a, b]$.

In a series of papers, Dragomir et al [5]-[8] proved (1.3) and some of its variants for $f' \in L_p[a, b]$ when $p \geq 1$, for Lebesgue norms making use of a peano kernel.

If we assume that $f' \in L_\infty[a, b]$ and $\|f'\|_\infty = \operatorname{ess\,sup}_{t \in [a, b]} |f'(t)|$ then M in (1.3) may be replaced by $\|f'\|_\infty$.

Dragomir et al [5]-[8] utilizing an integration by parts argument, obtained

$$|S(f; a, b)| \quad (1.4)$$

$$\leq \begin{cases} \frac{1}{b-a} \left[\left(\frac{b-a}{2} \right)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \|f'\|_\infty, & f' \in L_\infty[a, b], \\ \frac{1}{b-a} \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \|f'\|_p, & f' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{b-a} \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] \|f'\|_1, & f' \in L_1[a, b], \end{cases}$$

where $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$ and the constants $\frac{1}{4}$, $\left[\frac{1}{q+1} \right]^{\frac{1}{q}}$ and $\frac{1}{2}$ are sharp. In [14], Pachpatte established Čebyšev type inequalities by using Pecarić's extension of the Montgomery identity [17]. Cerone [1], proved the following inequality:

Lemma 1. Let $f : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous function. Define

$$\tau(x; \alpha, \beta) := f(x) - \frac{1}{\alpha + \beta} [\alpha M(f; a, x) + \beta M(f; x, b)], \quad (1.5)$$

where

$$|\tau(x; \alpha, \beta)| \quad (1.6)$$

$$\leq \begin{cases} \frac{1}{2(\alpha + \beta)} [\alpha(x-a) + \beta(b-x)] \|f'\|_\infty, & f' \in L_\infty[a, b], \\ \frac{1}{(\alpha + \beta)(q+1)^{\frac{1}{q}}} [\alpha^q(x-a) + \beta^q(b-x)]^{\frac{1}{q}} \|f'\|_p, & f' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{2} \left(1 + \frac{|\alpha - \beta|}{\alpha + \beta} \right) \|f'\|_1, & f' \in L_1[a, b], \end{cases}$$

where the usual L_p norms $\|k\|_p$ defined for a function $k \in L_p[a, b]$ as follows:

$$\|k\|_\infty := \operatorname{ess\,sup}_{t \in [a, b]} |k(t)|$$

and

$$\|k\|_p := \left(\int_a^b |k(t)|^p dt \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

With the help of two different kernels (1.7) and (1.9) given below, we extended the version of Cerone [1] and Dragomir's result [5]-[8].

Lemma 2. Let $P(x, \cdot) : [a, b] \rightarrow \mathbb{R}$, the peano type kernel is given by

$$p(x, t) = \begin{cases} \frac{\alpha}{\alpha+\beta} \frac{1}{x-a} [t - (a + h \frac{b-a}{2})], & a \leq t \leq x, \\ \frac{\beta}{\alpha+\beta} \frac{1}{b-x} [t - (b - h \frac{b-a}{2})], & x < t \leq b, \end{cases} \quad (1.7)$$

Then,

$$\begin{aligned} |\tau(x; \alpha, \beta)| &= \frac{1}{\alpha + \beta} \left[\begin{aligned} &\frac{\alpha}{x-a} \left\{ x - \left(a + h \frac{b-a}{2} \right) \right\} \\ &-\frac{\beta}{b-x} \left\{ x - \left(b - h \frac{b-a}{2} \right) \right\} \end{aligned} \right] f(x) \\ &+ \frac{h}{\alpha + \beta} \left(\frac{b-a}{2} \right) \left(\frac{\alpha}{x-a} f(a) + \frac{\beta}{b-x} f(b) \right) \\ &- \frac{1}{\alpha + \beta} \left[\frac{\alpha}{x-a} \int_a^x f(t) dt + \frac{\beta}{b-x} \int_x^b f(t) dt \right] \\ &\leq \begin{cases} \left(\begin{aligned} &\frac{\alpha}{x-a} \left\{ \frac{(x-a)^2}{4} + \left[\left(a + h \frac{b-a}{2} \right) - \frac{a+x}{2} \right]^2 \right\} \\ &+\frac{\beta}{b-x} \left\{ \frac{(b-x)^2}{4} + \left[\left(b - h \frac{b-a}{2} \right) - \frac{x+b}{2} \right]^2 \right\} \end{aligned} \right) \frac{1}{(\alpha+\beta)} \|f'\|_{\infty} \\ , \quad f' \in L_{\infty}[a, b] \end{cases} \\ &\leq \begin{cases} \left[\begin{aligned} &\frac{\alpha^q}{(x-a)^q} \left\{ \left(x - \left(a + h \frac{b-a}{2} \right) \right)^{q+1} - \left(h \frac{a-b}{2} \right)^{q+1} \right\} \\ &+\frac{\beta^q}{(b-x)^q} \left\{ \left(b - \left(x + h \frac{b-a}{2} \right) \right)^{q+1} - \left(h \frac{a-b}{2} \right)^{q+1} \right\} \end{aligned} \right]^{\frac{1}{q}} \frac{1}{(q+1)^{\frac{1}{q}} (\alpha+\beta)} \|f'\|_p, \\ f' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases} \\ &\leq \begin{cases} \left(\begin{aligned} &(\alpha + \beta) - h \frac{b-a}{2} \left[\frac{\alpha(b-x) + \beta(x-a)}{(x-a)(b-x)} \right] \\ &+ \left| (\alpha - \beta) + h \frac{b-a}{2} \left[\frac{\beta(x-a) - \alpha(b-x)}{(x-a)(b-x)} \right] \right| \end{aligned} \right) \frac{\|f'\|_1}{2(\alpha+\beta)}. \end{cases} \end{aligned}$$

Lemma 3. Denote by $P(x, \cdot) : [a, b] \rightarrow \mathbb{R}$ the kernel is given by

$$P(x, t) := \begin{cases} \frac{\alpha}{2(\alpha+\beta)(x-a)} (t-a)^2, & a \leq t \leq x, \\ \frac{\beta}{2(\alpha+\beta)(b-x)} (t-b)^2, & x < t \leq b, \end{cases} \quad (1.9)$$

Then,

$$\begin{aligned}
& |\tau(x; \alpha, \beta)| \\
&= \frac{1}{2(\alpha + \beta)} [\alpha(x - a) - \beta(b - x)] f'(x) - f(x) \\
&\quad + \frac{1}{\alpha + \beta} [\alpha M(f; a, x) + \beta M(f; x, b)], \\
&\leq \begin{cases} \left[\alpha(x - a)^2 + \beta(b - x)^2 \right] \frac{\|f''\|_\infty}{6(\alpha + \beta)}, f'' \in L_\infty[a, b] \\ \frac{1}{(2q+1)^{\frac{1}{q}}} \left[\alpha^q (x - a)^{q+1} + \beta^q (b - x)^{q+1} \right]^{\frac{1}{q}} \frac{\|f''\|_p}{2(\alpha + \beta)}, \\ f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ (\alpha(x - a) + \beta(b - x) + |\alpha(x - a) - \beta(b - x)|) \frac{\|f''\|_1}{4(\alpha + \beta)}, \\ f'' \in L_1[a, b]. \end{cases} \tag{1.10}
\end{aligned}$$

Using a generalized form of (1.9), we constructed a number of new results for twice differentiable functions. These results are given in Lemma 4 and theorem 2 which are more generalized by (1.8)-(1.10). These generalized inequalities will have applications in approximation theory, probability theory and numerical analysis. We will show in our paper an application of the obtained inequalities for cumulative distribution function.

2. MAIN RESULTS

We will start our main result with this lemma.

Lemma 4. Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous mapping. Denote by $P(x, \cdot) : [a, b] \rightarrow \mathbb{R}$ the kernel $P(x, t)$ is given by

$$P(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \frac{1}{x - a} \frac{1}{2} [t - (a + h \frac{b - a}{2})]^2, & a \leq t \leq x, \\ \frac{\beta}{\alpha + \beta} \frac{1}{b - x} \frac{1}{2} [t - (b - h \frac{b - a}{2})]^2, & x < t \leq b, \end{cases} \tag{2.1}$$

for all $x \in [a + h \frac{b - a}{2}, b - h \frac{b - a}{2}]$ and $h \in [0, 1]$, where $\alpha, \beta \in \mathbb{R}$ are non negative and not both zero. Before we state and prove our main theorem, we will prove the following identity:

$$\begin{aligned}
\int_a^b P(x, t) f''(t) dt &= \frac{1}{2(\alpha + \beta)} \left[\frac{\alpha}{x - a} \left(x - \left(a + h \frac{b - a}{2} \right) \right)^2 \right. \\
&\quad \left. - \frac{\beta}{b - x} \left(x - \left(b - h \frac{b - a}{2} \right) \right)^2 \right] f'(x) \\
&\quad - \frac{1}{(\alpha + \beta)} \left[\frac{\alpha}{x - a} \left(x - \left(a + h \frac{b - a}{2} \right) \right) \right. \\
&\quad \left. - \frac{\beta}{b - x} \left(x - \left(b - h \frac{b - a}{2} \right) \right) \right] f(x) \tag{2.2}
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{\alpha+\beta}h\frac{b-a}{2}\left[\frac{\alpha}{x-a}f(a)+\frac{\beta}{b-x}f(b)\right] \\
& +\frac{1}{\alpha+\beta}h^2\frac{(b-a)^2}{8}\left[\frac{\beta}{b-x}f'(b)-\frac{\alpha}{x-a}f'(a)\right] \\
& +\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^xf(t)dt+\frac{\beta}{b-x}\int_x^bf(t)dt\right].
\end{aligned}$$

Proof. From (2.1), we have

$$\begin{aligned}
\int_a^b P(x,t)f''(t)dt &= \frac{\alpha}{(\alpha+\beta)}\frac{1}{x-a}\int_a^x\frac{[t-(a+h\frac{b-a}{2})]^2}{2}f''(t)dt \\
&+ \frac{\beta}{(\alpha+\beta)}\frac{1}{b-x}\int_x^b\frac{[t-(b-h\frac{b-a}{2})]^2}{2}f''(t)dt.
\end{aligned}$$

After simplification, we get the required identity (2.2). \square

We now give our main theorem.

Theorem 2. Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous mapping. Define

$$\begin{aligned}
\tau(x; \alpha, \beta) : &= \frac{1}{2(\alpha+\beta)}\left[\frac{\alpha}{x-a}\left(x-\left(a+h\frac{b-a}{2}\right)\right)^2\right. \\
& \left.-\frac{\beta}{b-x}\left(x-\left(b-h\frac{b-a}{2}\right)\right)^2\right]f'(x) \\
& -\frac{1}{(\alpha+\beta)}\left[\frac{\alpha}{x-a}\left(x-\left(a+h\frac{b-a}{2}\right)\right)\right. \\
& \left.-\frac{\beta}{b-x}\left(x-\left(b-h\frac{b-a}{2}\right)\right)\right]f(x) \\
& -\frac{1}{\alpha+\beta}h\frac{b-a}{2}\left[\frac{\alpha}{x-a}f(a)+\frac{\beta}{b-x}f(b)\right] \\
& +\frac{1}{\alpha+\beta}h^2\frac{(b-a)^2}{8}\left[\frac{\beta}{b-x}f'(b)-\frac{\alpha}{x-a}f'(a)\right] \\
& +\frac{1}{\alpha+\beta}[\alpha M(f; a, x)+\beta M(f; x, b)],
\end{aligned} \tag{2.3}$$

where $M(f; a, b)$ is the integral mean defined in (1.2), then

$$|\tau(x; \alpha, \beta)| \quad (2.4)$$

$$\leq \begin{cases} \left[\frac{\alpha}{x-a} \left\{ \left[x - \left(a + h \frac{b-a}{2} \right) \right]^3 + \left(h \frac{b-a}{2} \right)^3 \right\} - \frac{\beta}{b-x} \left\{ \left(h \frac{b-a}{2} \right)^3 + \left[x - \left(b - h \frac{b-a}{2} \right) \right]^3 \right\} \right] \frac{\|f''\|_{\infty}}{6(\alpha+\beta)}, & f'' \in L_{\infty}[a, b], \\ \left[\frac{\alpha^q}{(x-a)^q} \left\{ \left[x - \left(a + h \frac{b-a}{2} \right) \right]^{2q+1} - \left(h \frac{a-b}{2} \right)^{2q+1} \right\} + \frac{\beta^q}{(b-x)^q} \left\{ \left(h \frac{a-b}{2} \right)^{2q+1} - \left[x - \left(b - h \frac{b-a}{2} \right) \right]^{2q+1} \right\} \right]^{\frac{1}{q}} \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}(\alpha+\beta)}, & f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left\{ \begin{aligned} & \alpha(x-a) + \beta(b-x) - h(b-a)(\alpha+\beta) \\ & + \frac{h^2(b-a)}{2} \left[\frac{\alpha}{x-a} + \frac{\beta}{b-x} \right] \\ & + \left| \beta(b-x) - \alpha(x-a) + h(b-a)(\alpha-\beta) \right. \\ & \left. + \frac{h^2(b-a)}{2} \left[\frac{\beta}{b-x} - \frac{\alpha}{x-a} \right] \right| \end{aligned} \right\} \frac{\|f''\|_1}{4(\alpha+\beta)}, & f'' \in L_1[a, b] \end{cases}$$

for all $x \in [a, b]$, where $\|k\|$ is the usual Lebesgue norm for $k \in L[a, b]$ with

$$\|k\|_{\infty} := \operatorname{ess\,sup}_{t \in [a, b]} |k(t)| < \infty$$

and

$$\|k\|_p := \left(\int_a^b |k(t)|^p dt \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

Proof. Taking the modulus of (2.2) and using (2.3) and (1.2), we have

$$|\tau(x; \alpha, \beta)| = \left| \int_a^b P(x, t) f''(t) dt \right| \leq \int_a^b |P(x, t)| |f''(t)| dt. \quad (2.5)$$

Therefore, for $f'' \in L_{\infty}[a, b]$ we obtain

$$|\tau(x; \alpha, \beta)| \leq \|f''\|_{\infty} \int_a^b |P(x, t)| dt.$$

Now let us observe that

$$\begin{aligned} & \int_a^b |P(x, t)| dt \\ &= \frac{\alpha}{2(\alpha+\beta)(x-a)} \int_a^x \left[t - \left(a + h \frac{b-a}{2} \right) \right]^2 dt \\ & \quad + \frac{\beta}{2(\alpha+\beta)(b-x)} \int_x^b \left[t - \left(b - h \frac{b-a}{2} \right) \right]^2 dt. \end{aligned}$$

After simple integration, we get

$$\begin{aligned} & \int_a^b |P(x, t)| dt \\ &= \frac{1}{6(\alpha + \beta)} \left[\begin{aligned} & \frac{\alpha}{x-a} \left\{ \left[x - \left(a + h \frac{b-a}{2} \right) \right]^3 + \left(h \frac{b-a}{2} \right)^3 \right\} \\ & - \frac{\beta}{b-x} \left\{ \left(h \frac{b-a}{2} \right)^3 - \left[x - \left(b - h \frac{b-a}{2} \right) \right]^3 \right\} \end{aligned} \right]. \end{aligned}$$

Hence the first inequality is obtained.

$$\begin{aligned} & |\tau(x; \alpha, \beta)| \\ & \leq \left[\begin{aligned} & \frac{\alpha}{x-a} \left\{ \left[x - \left(a + h \frac{b-a}{2} \right) \right]^3 + \left(h \frac{b-a}{2} \right)^3 \right\} \\ & - \frac{\beta}{b-x} \left\{ \left(h \frac{b-a}{2} \right)^3 + \left[x - \left(b - h \frac{b-a}{2} \right) \right]^3 \right\} \end{aligned} \right] \frac{1}{6(\alpha + \beta)} \|f''\|_{\infty}. \end{aligned}$$

Further, using Hölder's integral inequality in (2.5) we have for $f'' \in L_p[a, b]$

$$|\tau(x; \alpha, \beta)| \leq \|f''\|_p \left(\int_a^b |P(x, t)|^q dt \right)^{\frac{1}{q}}.$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. Now

$$\begin{aligned} & (\alpha + \beta) \left(\int_a^b |P(x, t)|^q dt \right)^{\frac{1}{q}} \\ &= \left[\begin{aligned} & \frac{\alpha^q}{2^q(x-a)^q} \int_a^x \left[t - \left(a + h \frac{b-a}{2} \right) \right]^{2q} dt \\ & + \frac{\beta^q}{2^q(b-x)^q} \int_x^b \left[t - \left(b - h \frac{b-a}{2} \right) \right]^{2q} dt \end{aligned} \right]^{\frac{1}{q}} \\ &= \left[\begin{aligned} & \frac{\alpha^q}{2^q(2q+1)(x-a)^q} \left[t - \left(a + h \frac{b-a}{2} \right) \right]^{2q+1} \Big|_a^x \\ & + \frac{\beta^q}{2^q(2q+1)(b-x)^q} \left[t - \left(b - h \frac{b-a}{2} \right) \right]^{2q} \Big|_x^b \end{aligned} \right]^{\frac{1}{q}}. \end{aligned}$$

Again, after simple integration, we get

$$\begin{aligned} & (\alpha + \beta) \left(\int_a^b |P(x, t)|^q dt \right)^{\frac{1}{q}} \\ &= \frac{1}{2(2q+1)^{\frac{1}{q}}} \left[\begin{aligned} & \frac{\alpha^q}{(x-a)^q} \left\{ \left[x - \left(a + h \frac{b-a}{2} \right) \right]^{2q+1} - \left(h \frac{b-a}{2} \right)^{2q+1} \right\} \\ & + \frac{\beta^q}{(b-x)^q} \left\{ \left(h \frac{b-a}{2} \right)^{2q+1} - \left[x - \left(b - h \frac{b-a}{2} \right) \right]^{2q+1} \right\} \end{aligned} \right]^{\frac{1}{q}}. \end{aligned}$$

Hence the second inequality is obtained as below.

$$|\tau(x; \alpha, \beta)| \leq \frac{1}{2(2q+1)^{\frac{1}{q}}} \left[\frac{\alpha^q}{(x-a)^q} \left\{ \left[x - \left(a + h \frac{b-a}{2} \right) \right]^{2q+1} - \left(h \frac{a-b}{2} \right)^{2q+1} \right\} + \frac{\beta^q}{(b-x)^q} \left\{ \left(h \frac{a-b}{2} \right)^{2q+1} - \left[x - \left(b - h \frac{b-a}{2} \right) \right]^{2q+1} \right\} \right]^{\frac{1}{q}} \|f''\|_p.$$

Finally, for $f'' \in L_1[a, b]$, using (2.1), we have the following inequality from (2.5),

$$|\tau(x; \alpha, \beta)| \leq \sup_{t \in [a, b]} |P(x, t)| \|f''\|_1,$$

where

$$\begin{aligned} (\alpha + \beta) \sup_{t \in [a, b]} |P(x, t)| &= \frac{1}{4} \left\{ \begin{aligned} &\alpha(x-a) + \beta(b-x) - h(b-a)(\alpha + \beta) \\ &+ \frac{h^2(b-a)}{2} \left[\frac{\alpha}{x-a} + \frac{\beta}{b-x} \right] \end{aligned} \right\} \\ &\quad + \frac{1}{4} |\beta(b-x) - \alpha(x-a) + h(b-a)(\alpha - \beta) \\ &\quad + \frac{h^2(b-a)}{2} \left[\frac{\beta}{b-x} - \frac{\alpha}{x-a} \right]| \end{aligned}$$

This gives us the last inequality as below.

$$|\tau(x; \alpha, \beta)| \leq \left\{ \begin{aligned} &\alpha(x-a) + \beta(b-x) - h(b-a)(\alpha + \beta) \\ &+ \frac{h^2(b-a)}{2} \left[\frac{\alpha}{x-a} + \frac{\beta}{b-x} \right] \\ &+ |\beta(b-x) - \alpha(x-a) \\ &+ h(b-a)(\alpha - \beta) + \frac{h^2(b-a)}{2} \left[\frac{\beta}{b-x} - \frac{\alpha}{x-a} \right]| \end{aligned} \right\} \frac{\|f''\|_1}{4(\alpha + \beta)}.$$

This completes the proof of theorem. □

Some special cases of Theorem 2

In this section, we will give some useful cases.

Remark 1. If we put $h = 0$ in (2.4), we get (1.10).

Remark 2. If we put $h = 0$ in (1.8), we get Cerone's result given in (1.6).

Remark 3. If we put $h = 1$ in (2.4), we get a new result.

$$\begin{aligned}
 & |\tau(x; \alpha, \beta)| \\
 & \leq \begin{cases} \left[\begin{array}{l} \frac{\alpha}{x-a} \left\{ (x-A)^3 + \left(\frac{b-a}{2}\right)^3 \right\} \\ - \frac{\beta}{b-x} \left\{ \left(\frac{b-a}{2}\right)^3 + (x-A)^3 \right\} \end{array} \right] \frac{1}{6(\alpha+\beta)} \|f''\|_{\infty}, & f'' \in L_{\infty}[a, b], \\ \\ \left[\begin{array}{l} \frac{1}{(x-a)^q} \left\{ (x-A)^{2q+1} - \left(\frac{a-b}{2}\right)^{2q+1} \right\} \\ + \frac{1}{(b-x)^q} \left\{ \left(\frac{a-b}{2}\right)^{2q+1} - (x-A)^{2q+1} \right\} \end{array} \right]^{\frac{1}{q}} \frac{1}{(2q+1)^{\frac{1}{q}}} \frac{1}{4} \|f''\|_p, & f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \left\{ \begin{array}{l} \alpha(x-a) + \beta(b-x) - (b-a)(\alpha+\beta) \\ + \frac{b-a}{2} \left[\frac{\alpha}{x-a} + \frac{\beta}{b-x} \right] \\ + |\beta(b-x) - \alpha(x-a) + (b-a)(\alpha-\beta) \\ + \frac{b-a}{2} \left[\frac{\beta}{b-x} - \frac{\alpha}{x-a} \right]| \end{array} \right\} \frac{\|f''\|_1}{4(\alpha+\beta)}, & f'' \in L_1[a, b], \end{cases}
 \end{aligned} \tag{2.6}$$

where $A = \frac{a+b}{2}$.

Corollary 1. If we put $x = A$ in above, we get

$$\begin{aligned}
 & |\tau(A; \alpha, \beta)| \\
 & \leq \begin{cases} \frac{(b-a)^2}{24} \frac{\alpha-\beta}{\alpha+\beta} \|f''\|_{\infty}, & f'' \in L_{\infty}[a, b], \\ \\ \left[\frac{\beta^q}{(b-a)^q} \left(\frac{a-b}{2}\right)^{2q+1} - \frac{\alpha^q}{(b-a)^q} \left(\frac{a-b}{2}\right)^{2q+1} \right]^{\frac{1}{q}} \frac{1}{(2q+1)^{\frac{1}{q}}} \frac{1}{(\alpha+\beta)} \|f''\|_p, & f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \left\{ \left(1 - \frac{a-b}{2}\right) + \left| \frac{\beta-\alpha}{\alpha+\beta} \left(1 - \frac{a-b}{2}\right) \right| \right\} \frac{\|f''\|_1}{4}, & f'' \in L_1[a, b]. \end{cases}
 \end{aligned} \tag{2.7}$$

Remark 4. If we put $h = \frac{1}{2}$ in (2.3) and (2.4) we get the following inequality:

$$\begin{aligned}
 & \left| \frac{1}{2(\alpha+\beta)} \left[\frac{\alpha}{x-a} \left(x - \frac{3a+b}{4} \right)^2 - \frac{\beta}{b-x} \left(x - \frac{a+3b}{4} \right)^2 \right] f'(x) \right. \\
 & \quad - \frac{1}{(\alpha+\beta)} \left[\frac{\alpha}{x-a} \left(x - \frac{3a+b}{4} \right) - \frac{\beta}{b-x} \left(x - \frac{a+3b}{4} \right) \right] f(x) \\
 & \quad - \frac{1}{\alpha+\beta} \frac{b-a}{4} \left[\frac{\alpha}{x-a} f(a) + \frac{\beta}{b-x} f(b) \right] \\
 & \quad + \frac{1}{\alpha+\beta} \frac{(b-a)^2}{32} \left[\frac{\beta}{b-x} f'(b) - \frac{\alpha}{x-a} f'(a) \right] \\
 & \quad \left. + \frac{1}{\alpha+\beta} [\alpha M(f; a, x) + \beta M(f; x, b)] \right| \tag{2.8} \\
 & \leq \begin{cases} \left[\frac{\alpha}{x-a} \left\{ \left[x - \left(a + \frac{b-a}{4} \right) \right]^3 + \left(\frac{b-a}{4} \right)^3 \right\} \right. \\ \quad \left. - \frac{\beta}{b-x} \left\{ \left(\frac{b-a}{4} \right)^3 + \left[x - \left(b - \frac{b-a}{4} \right) \right]^3 \right\} \right] \frac{\|f''\|_{\infty}}{6(\alpha+\beta)}, & f'' \in L_{\infty}[a, b], \\ \\ \left[\frac{\alpha^q}{(x-a)^q} \left\{ \left[x - \left(a + \frac{b-a}{4} \right) \right]^{2q+1} - \left(\frac{a-b}{4} \right)^{2q+1} \right\} \right. \\ \quad \left. + \frac{\beta^q}{(b-x)^q} \left\{ \left(\frac{a-b}{4} \right)^{2q+1} - \left[x - \left(b - \frac{b-a}{4} \right) \right]^{2q+1} \right\} \right]^{\frac{1}{q}} \frac{\|f''\|_p}{2(2q+1)^{\frac{1}{q}}(\alpha+\beta)}, & f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \left\{ \begin{aligned} & \alpha(x-a) + \beta(b-x) - \frac{1}{2}(b-a)(\alpha+\beta) \\ & + \frac{(b-a)}{8} \left[\frac{\alpha}{x-a} + \frac{\beta}{b-x} \right] \\ & + \left| \beta(b-x) - \alpha(x-a) + \frac{1}{2}(b-a)(\alpha-\beta) \right| \\ & + \frac{(b-a)}{8} \left[\frac{\beta}{b-x} - \frac{\alpha}{x-a} \right] \end{aligned} \right\} \frac{\|f''\|_1}{4(\alpha+\beta)}, & f'' \in L_1[a, b]. \end{cases}
 \end{aligned}$$

Corollary 2. If we put $\alpha = \beta$ and $x = A$ in (2.8) we get another result.

$$\begin{aligned}
 & \left| \frac{1}{2} f\left(\frac{a+b}{2}\right) + \frac{1}{4} [f(a) + f(b)] \right. \\
 & \quad \left. - \frac{b-a}{32} [f'(b) - f'(a)] - (b-a) \int_a^b f(t) dt \right| \tag{2.9} \\
 & \leq \begin{cases} \frac{(b-a)^2 \|f''\|_{\infty}}{192}, & f'' \in L_{\infty}[a, b], \\ \\ \left[\left\{ (b-a)^{2q+1} - (a-b)^{2q+1} \right\} \right]^{\frac{1}{q}} \frac{\|f''\|_p}{16(b-a)2^{2\frac{1}{q}}(2q+1)^{\frac{1}{q}}}, & f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \frac{(b-a) \|f''\|_1}{16}, & f'' \in L_1[a, b]. \end{cases}
 \end{aligned}$$

Corollary 3. *If we put $\alpha = \beta$ and $x = \frac{a+3b}{4}$ in (2.8) we get another new result.*

$$\begin{aligned}
 & \left| \frac{b-a}{12} f'(x) - \left(\frac{1}{3} - \frac{2}{b-a} \right) f(x) - \frac{1}{2} \left(\frac{1}{3} f(a) + f(b) \right) \right. \\
 & \quad \left. + \frac{b-a}{16} (f'(b) - \frac{1}{3} f'(a)) \right. \\
 & \quad \left. + \frac{2}{b-a} \left(\frac{1}{3} \int_a^{\frac{a+3b}{4}} f(t) dt + \int_{\frac{a+3b}{4}}^b f(t) dt \right) \right| \\
 & \leq \begin{cases} \frac{(b-a)^2 \|f''\|_\infty}{96}, & f'' \in L_\infty[a, b], \\ \left[\frac{1}{3^q} \left\{ \left(\frac{b-a}{2} \right)^{2q+1} - \left(\frac{a-b}{4} \right)^{2q+1} \right\} + \left(\frac{a-b}{4} \right)^{2q+1} \right]^{\frac{1}{q}} \frac{\|f''\|_p}{(b-a)(2q+1)^{\frac{1}{q}}}, & f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left\{ \frac{2}{3} + \left| \frac{b-a}{2} - \frac{1}{3} \right| \right\} \frac{\|f''\|_1}{8}, & f'' \in L_1[a, b]. \end{cases} \quad (2.10)
 \end{aligned}$$

Hence, for different values of h , we can obtain a variety of results.

Remark 5. *We can write (2.3) in another way. Since*

$$\begin{aligned}
 & \alpha M(f; a, x) + \beta M(f; x, b) \\
 & = \alpha M(f; a, x) + \frac{\beta}{b-x} \left(\int_a^b f(u) du - \int_a^x f(u) du \right).
 \end{aligned}$$

or

$$\begin{aligned}
 & \alpha M(f; a, x) + \beta M(f; x, b) \\
 & = \alpha M(f; a, x) - \frac{\beta}{b-x} \int_a^x f(u) du + \frac{\beta}{b-x} \int_a^b f(u) du \\
 & = (\alpha + \beta - \beta \sigma(x)) M(f; a, x) + \beta \sigma(x) M(f; a, b),
 \end{aligned}$$

where

$$\frac{b-a}{b-x} = \sigma(x). \quad (2.11)$$

Thus, from (2.3),

$$\begin{aligned}
& \tau(x; \alpha, \beta) \\
&= \frac{1}{2(\alpha+\beta)} \left[\frac{\alpha}{x-a} \left(x - \left(a + h \frac{b-a}{2} \right) \right)^2 \right. \\
&\quad \left. - \frac{\beta}{b-x} \left(x - \left(b - h \frac{b-a}{2} \right) \right)^2 \right] f'(x) \\
&\quad - \frac{1}{(\alpha+\beta)} \left[\frac{\alpha}{x-a} \left(x - \left(a + h \frac{b-a}{2} \right) \right) \right. \\
&\quad \left. - \frac{\beta}{b-x} \left(x - \left(b - h \frac{b-a}{2} \right) \right) \right] f(x) \\
&\quad - \frac{1}{\alpha+\beta} h \frac{b-a}{2} \left[\frac{\alpha}{x-a} f(a) + \frac{\beta}{b-x} f(b) \right] \\
&\quad + \frac{1}{\alpha+\beta} h^2 \frac{(b-a)^2}{8} \left[\frac{\beta}{b-x} f'(b) - \frac{\alpha}{x-a} f'(a) \right] \\
&\quad + \left[\left(1 - \frac{\beta}{\alpha+\beta} \sigma(x) \right) M(f; a, x) + \frac{\beta}{\alpha+\beta} \sigma(x) M(f; a, b) \right].
\end{aligned} \tag{2.12}$$

so that for fixed $[a, b]$, $M(f; a, b)$ is also fixed.

Corollary 4. If (2.3) and (2.4) is evaluated at $x = \frac{a+b}{2}$ and $\alpha = \beta$ then

$$\begin{aligned}
& \left| (h-1) f\left(\frac{a+b}{2}\right) - \frac{h}{2} (f(a) + f(b)) \right. \\
& \left. + h^2 \frac{b-a}{8} (f'(b) - f'(a)) + \frac{1}{b-a} \int_a^b f(t) dt \right| \\
& \leq \begin{cases} (1-h)^3 \frac{(b-a)^2}{24} \|f''\|_\infty, & f'' \in L_\infty[a, b], \\ \left[\frac{2^q}{(b-a)^q} \left\{ \left(\frac{b-a}{2} (1-h) \right)^{2q+1} - \left(h \frac{a-b}{2} \right)^{2q+1} \right\} \right. \\ \quad \left. + \frac{2^q}{(b-a)^q} \left\{ \left(h \frac{a-b}{2} \right)^{2q+1} - \left(\frac{b-a}{2} (1-h) \right)^{2q+1} \right\} \right] \frac{\|f''\|_p}{4(2q+1)^{\frac{1}{q}}} \\ \quad , f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left[(b-a)(1-2h) + 2h^2 \right] \frac{\|f''\|_1}{8}, & f'' \in L_1[a, b]. \end{cases}
\end{aligned} \tag{2.13}$$

3. Perturbed Results

In 1882, Čebyšev [4] gave the following inequality.

$$|T(f, g)| \leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty \tag{3.1}$$

where $f, g : [a, b] \rightarrow \mathbb{R}$ are absolutely continuous functions, which has bounded first derivatives such that

$$\begin{aligned}
T(f, g) &= \frac{1}{b-a} \int_a^b f(x) g(x) dx \\
&\quad - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \\
&= M(f, g; a, b) - M(f; a, b) M(g; a, b),
\end{aligned} \tag{3.2}$$

and $\|\cdot\|_\infty$ denotes the norm in $L_\infty[a, b]$ defined as $\|p\|_\infty = \text{ess sup}_{t \in [a, b]} |P(t)|$.

In 1935, Grüss [9] proved the following inequality:

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{b-a} \int_a^b f(x) dx \frac{1}{b-a} \int_a^b g(x) dx \right| \quad (3.3)$$

$$\leq \frac{1}{4} (\Phi - \varphi) (\Gamma - \gamma),$$

provided that f and g are two integrable functions on $[a, b]$ and satisfy the condition:

$$\varphi \leq f(x) \leq \Phi \quad \text{and} \quad \gamma \leq g(x) \leq \Gamma, \quad \text{for all } x \in [a, b]. \quad (3.4)$$

The constant $\frac{1}{4}$ is best possible. The perturbed version of the results of Theorem 2 can be obtained by using Grüss type results involving the Čebyšev functional.

$$T(f, g) = M(f, g; a, b) - M(f; a, b) M(g; a, b),$$

where M is the integral mean and is defined in (1.2).

Theorem 3. Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous mapping and α, β are non-negative real numbers, then

$$\left| \tau(x; \alpha, \beta) - \frac{1}{(\alpha + \beta)} \left[\frac{\alpha}{x-a} \left\{ \left[x - \left(a + h \frac{b-a}{2} \right) \right]^3 + \left(h \frac{b-a}{2} \right)^3 \right\} - \frac{\beta}{b-x} \left\{ \left(h \frac{b-a}{2} \right)^3 + \left[x - \left(b - h \frac{b-a}{2} \right) \right]^3 \right\} \right] \right| \frac{\kappa}{6} \quad (3.5)$$

$$\leq (b-a) N(x) \left[\frac{1}{b-a} \|f''\|_2^2 - \kappa^2 \right]^{\frac{1}{2}}$$

$$\leq (b-a) (\Gamma - \gamma) \lambda,$$

where, $\tau(x; \alpha, \beta)$ is as given by (2.3) and $\lambda = \Phi - \varphi$. Let

$$\kappa = \frac{f'(b) - f'(a)}{b-a} \quad (3.6)$$

then

$$N^2(x) = \frac{1}{20(\alpha + \beta)^2} \left\{ \frac{\alpha^2}{(x-a)^2} \left[\left(x - \left(a + h \frac{b-a}{2} \right) \right)^5 + \left(h \frac{b-a}{2} \right)^5 \right] + \frac{\beta^2}{(b-x)^2} \left[\left(h \frac{b-a}{2} \right)^5 - \left(x - \left(b - h \frac{b-a}{2} \right) \right)^5 \right] \right\} \quad (3.7)$$

$$- \left(\frac{1}{6(b-a)(\alpha + \beta)} \left[\frac{\alpha}{x-a} \left\{ \left[x - \left(a + h \frac{b-a}{2} \right) \right]^3 + \left(h \frac{b-a}{2} \right)^3 \right\} - \frac{\beta}{b-x} \left\{ \left(h \frac{b-a}{2} \right)^3 + \left[x - \left(b - h \frac{b-a}{2} \right) \right]^3 \right\} \right] \right)^2$$

Proof. Associating $f(t)$ with $P(x, t)$ and $g(t)$ with $f''(t)$, then from (2.1) and (3.2), we obtain

$$T(P(x, \cdot), f''(\cdot); a, b) = M(P(x, \cdot), f''(\cdot); a, b) - M(P(x, \cdot); a, b) M(f''(\cdot); a, b)$$

Now using identity (2.2),

$$(b-a) T(P(x, \cdot), f''(\cdot); a, b) = \tau(x; \alpha, \beta) - (b-a) M(P(x, \cdot); a, b) \kappa \quad (3.8)$$

where κ is the secant slope of f' over $[a, b]$, as given in (3.6). Now, from (2.2) and (3.2),

$$\begin{aligned}
 & (b-a) M(P(x, \cdot); a, b) \\
 = & \int_a^b P(x, t) dt \\
 = & \frac{\alpha}{2(\alpha + \beta)(x-a)} \int_a^x \left[t - \left(a + h \frac{b-a}{2} \right) \right]^2 dt \\
 & + \frac{\beta}{2(\alpha + \beta)(b-x)} \int_x^b \left[t - \left(b - h \frac{b-a}{2} \right) \right]^2 dt \\
 = & \frac{1}{6(\alpha + \beta)} \left[\begin{aligned} & \frac{\alpha}{x-a} \left\{ \left[x - \left(a + h \frac{b-a}{2} \right) \right]^3 + \left(h \frac{b-a}{2} \right)^3 \right\} \\ & - \frac{\beta}{b-x} \left\{ \left(h \frac{b-a}{2} \right)^3 + \left[x - \left(b - h \frac{b-a}{2} \right) \right]^3 \right\} \end{aligned} \right]
 \end{aligned} \tag{3.9}$$

Now combining (3.9) with (3.7) the left hand side of (3.5) is obtained.

Let $f, g : [a, b] \rightarrow \mathbb{R}$ and $fg : [a, b] \rightarrow \mathbb{R}$ be integrable on $[a, b]$, then [1]

$$\begin{aligned}
 |T(f, g)| & \leq T^{\frac{1}{2}}(f, f) T^{\frac{1}{2}}(g, g) & (f, g \in L_2[a, b]) \\
 & \leq \frac{(\Gamma - \gamma)}{2} T^{\frac{1}{2}}(f, f) & (\gamma \leq g(x) \leq \Gamma, t \in [a, b]) \\
 & \leq \frac{1}{4} (\Phi - \varphi) (\Gamma - \gamma) & (\varphi \leq f(x) \leq \Phi, t \in [a, b]).
 \end{aligned} \tag{3.10}$$

Also, note that

$$\begin{aligned}
 0 & \leq T^{\frac{1}{2}}(f''(\cdot), f''(\cdot)) \\
 & = [M(f''(\cdot)^2; a, b) - M^2(f''(\cdot); a, b)]^{\frac{1}{2}} \\
 & = \left[\frac{1}{b-a} \int_a^b \|f''(t)\|^2 dt - \left(\frac{\int_a^b f''(t) dt}{b-a} \right)^2 \right]^{\frac{1}{2}} \\
 & = \left[\frac{1}{b-a} \|f''\|_2^2 - \kappa^2 \right]^{\frac{1}{2}} \\
 & \leq \frac{(\Gamma - \gamma)}{2}
 \end{aligned} \tag{3.11}$$

where $\gamma \leq f''(t) \leq \Gamma, t \in [a, b]$. Now, for the bounds on (3.8), we have to determine $T^{\frac{1}{2}}(P(x, \cdot), P(x, \cdot))$ and $\varphi \leq P(x, \cdot) \leq \Phi$ from (3.10) and (3.11).

Now from (2.1), the definition of $P(x, t)$, we have

$$T(P(x, \cdot), P(x, \cdot)) = M(P^2(x, \cdot); a, b) - M^2(P(x, \cdot); a, b). \tag{3.12}$$

From (3.10) we obtain

$$\begin{aligned} & M(P(x, \cdot); a, b) \\ &= \frac{1}{6(\alpha + \beta)} \left[\begin{aligned} & \frac{\alpha}{x-a} \left\{ \left[x - \left(a + h \frac{b-a}{2} \right) \right]^3 + \left(h \frac{b-a}{2} \right)^3 \right\} \\ & - \frac{\beta}{b-x} \left\{ \left(h \frac{b-a}{2} \right)^3 + \left[x - \left(b - h \frac{b-a}{2} \right) \right]^3 \right\} \end{aligned} \right] \end{aligned}$$

and

$$\begin{aligned} & (b-a) M(P^2(x, \cdot); a, b) \\ &= \left(\frac{\alpha}{\alpha + \beta} \right)^2 \frac{1}{4(x-a)^2} \int_a^x \left[t - \left(a + h \frac{b-a}{2} \right) \right]^4 dt \\ & \quad + \left(\frac{\beta}{\alpha + \beta} \right)^2 \frac{1}{4(b-x)^2} \int_x^b \left[t - \left(b - h \frac{b-a}{2} \right) \right]^4 dt \\ &= \left(\frac{\alpha}{\alpha + \beta} \right)^2 \frac{1}{20(x-a)^2} \left[\left(x - \left(a + h \frac{b-a}{2} \right) \right)^5 + \left(h \frac{b-a}{2} \right)^5 \right] \\ & \quad + \left(\frac{\beta}{\alpha + \beta} \right)^2 \frac{1}{20(b-x)^2} \left[\left(h \frac{b-a}{2} \right)^5 - \left(x - \left(b - h \frac{b-a}{2} \right) \right)^5 \right] \\ &= \frac{1}{20(\alpha + \beta)^2} \left\{ \begin{aligned} & \frac{\alpha^2}{(x-a)^2} \left[\left(x - \left(a + h \frac{b-a}{2} \right) \right)^5 + \left(h \frac{b-a}{2} \right)^5 \right] \\ & + \frac{\beta^2}{(b-x)^2} \left[\left(h \frac{b-a}{2} \right)^5 - \left(x - \left(b - h \frac{b-a}{2} \right) \right)^5 \right] \end{aligned} \right\} \end{aligned}$$

Thus, substituting the above results into (3.12) gives

$$0 \leq N(x) = T^{\frac{1}{2}}(P(x, \cdot), P(x, \cdot))$$

which is given explicitly by (3.7). Combining (3.8), (3.12) and (3.11) give from the first inequality in (3.10), the first inequality in (3.5). Now utilizing the inequality in (3.11) produces the second result in (3.5). Further, it may be noticed from the definition of $P(x, t)$ in (2.1) that for $\alpha, \beta \geq 0$, give

$$\begin{aligned} \Phi &= \sup_{t \in [a, b]} P(x, t) \\ &= \frac{1}{4(\alpha + \beta)} \left\{ \begin{aligned} & \alpha(x-a) + \beta(b-x) - h(b-a)(\alpha + \beta) \\ & + \frac{h^2(b-a)}{2} \left[\frac{\alpha}{x-a} + \frac{\beta}{b-x} \right] \\ & + \left| \beta(b-x) - \alpha(x-a) + h(b-a)(\alpha - \beta) \right. \\ & \quad \left. + \frac{h^2(b-a)}{2} \left[\frac{\beta}{b-x} - \frac{\alpha}{x-a} \right] \right| \end{aligned} \right\} \\ \varphi &= \inf_{t \in [a, b]} P(x, t) \\ &= \frac{h^2(b-a)^2}{8(\alpha + \beta)} \left\{ \frac{\alpha}{x-a} + \frac{\beta}{b-x} - \left| \frac{\alpha}{x-a} - \frac{\beta}{b-x} \right| \right\} \end{aligned}$$

where $\Phi - \varphi = \lambda$.

□

4. An Application to the Cumulative Distribution Function

Let $X \in [a, b]$ be a random variable with the cumulative distributive function

$$F(x) = P_r(X \leq x) = \int_a^x f(u) du,$$

where f is the probability density function. In particular,

$$\int_a^b f(u) du = 1.$$

The following theorem holds.

Theorem 4. *Let X and F be as above, then*

$$\begin{aligned} & \left| \frac{1}{2} \left[\alpha(b-x) \left(x - \left(a + h \frac{b-a}{2} \right) \right)^2 - \beta(x-a) \left(x - \left(b - h \frac{b-a}{2} \right) \right)^2 \right] f'(x) \right. \\ & \quad - \left[\alpha(b-x) \left(x - \left(a + h \frac{b-a}{2} \right) \right) - \beta(x-a) \left(x - \left(b - h \frac{b-a}{2} \right) \right) \right] f(x) \\ & \quad - h \frac{b-a}{2} [\alpha(b-x)f(a) + \beta(x-a)f(b)] \\ & \quad + h^2 \frac{(b-a)^2}{8} [\beta(x-a)f'(b) - \alpha(b-x)f'(a)] \\ & \quad \left. + [\alpha(b-x) - \beta(x-a)] F(x) + \beta(x-a) \right| \\ & \leq \begin{cases} \frac{\|f''\|_\infty}{6} \left[\begin{array}{l} \alpha(b-x) \left\{ \left[x - \left(a + h \frac{b-a}{2} \right) \right]^3 + \left(h \frac{b-a}{2} \right)^3 \right\} \\ - \beta(x-a) \left\{ \left(h \frac{b-a}{2} \right)^3 + \left[x - \left(b - h \frac{b-a}{2} \right) \right]^3 \right\} \end{array} \right], f'' \in L_\infty[a, b], \\ \frac{(b-x)(x-a)\|f''\|_p}{2(2q+1)^{\frac{1}{q}}} \left[\begin{array}{l} \frac{\alpha^q}{(x-a)^q} \left\{ \left[x - \left(a + h \frac{b-a}{2} \right) \right]^{2q+1} - \left(h \frac{b-a}{2} \right)^{2q+1} \right\} \\ + \frac{\beta^q}{(b-x)^q} \left\{ \left(h \frac{b-a}{2} \right)^{2q+1} - \left[x - \left(b - h \frac{b-a}{2} \right) \right]^{2q+1} \right\} \end{array} \right]^{\frac{1}{q}}, \\ , f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{(b-x)(x-a)\|f''\|_1}{4} \left\{ \begin{array}{l} \alpha(x-a) + \beta(b-x) - h(b-a)(\alpha + \beta) \\ + \frac{h^2(b-a)}{2} \left[\frac{\alpha}{x-a} + \frac{\beta}{b-x} \right] \\ \beta(b-x) - \alpha(x-a) + h(b-a)(\alpha - \beta) \\ + \frac{h^2(b-a)}{2} \left[\frac{\beta}{b-x} - \frac{\alpha}{x-a} \right] \end{array} \right\} \end{cases}. \end{aligned} \tag{4.1}$$

Proof. From (2.3), and by using the definition of Probability Density Function, we have

$$\begin{aligned}
\tau(x; \alpha, \beta) &: = \frac{1}{2(\alpha + \beta)} \left[\frac{\alpha}{x-a} \left(x - \left(a + h \frac{b-a}{2} \right) \right)^2 - \frac{\beta}{b-x} \left(x - \left(b - h \frac{b-a}{2} \right) \right)^2 \right] f'(x) \\
&\quad - \frac{1}{(\alpha + \beta)} \left[\frac{\alpha}{x-a} \left(x - \left(a + h \frac{b-a}{2} \right) \right) - \frac{\beta}{b-x} \left(x - \left(b - h \frac{b-a}{2} \right) \right) \right] f(x) \\
&\quad - \frac{1}{\alpha + \beta} h \frac{b-a}{2} \left[\frac{\alpha}{x-a} f(a) + \frac{\beta}{b-x} f(b) \right] \\
&\quad + \frac{1}{\alpha + \beta} h^2 \frac{(b-a)^2}{8} \left[\frac{\beta}{b-x} f'(b) - \frac{\alpha}{x-a} f'(a) \right] \\
&\quad + \frac{1}{\alpha + \beta} [\alpha M(f; a, x) + \beta M(f; x, b)], \\
&= \frac{1}{2(\alpha + \beta)} \left[\frac{\alpha}{x-a} \left(x - \left(a + h \frac{b-a}{2} \right) \right)^2 - \frac{\beta}{b-x} \left(x - \left(b - h \frac{b-a}{2} \right) \right)^2 \right] f'(x) \\
&\quad - \frac{1}{(\alpha + \beta)} \left[\frac{\alpha}{x-a} \left(x - \left(a + h \frac{b-a}{2} \right) \right) - \frac{\beta}{b-x} \left(x - \left(b - h \frac{b-a}{2} \right) \right) \right] f(x) \\
&\quad - \frac{1}{\alpha + \beta} h \frac{b-a}{2} \left[\frac{\alpha}{x-a} f(a) + \frac{\beta}{b-x} f(b) \right] \\
&\quad + \frac{1}{\alpha + \beta} h^2 \frac{(b-a)^2}{8} \left[\frac{\beta}{b-x} f'(b) - \frac{\alpha}{x-a} f'(a) \right] \\
&\quad + \frac{1}{\alpha + \beta} \left\{ \left[\frac{\alpha(b-x) - \beta(x-a)}{(x-a)(b-x)} \right] F(x) + \frac{\beta}{(b-x)} \right\}
\end{aligned}$$

or

$$\begin{aligned}
&(\alpha + \beta)(x-a)(b-x)\tau(x; \alpha, \beta) \tag{4.2} \\
&= \frac{1}{2} \left[\frac{\alpha(b-x)(x - (a + h \frac{b-a}{2}))^2}{-\beta(x-a)(x - (b - h \frac{b-a}{2}))^2} \right] f'(x) \\
&\quad - \left[\frac{\alpha(b-x)(x - (a + h \frac{b-a}{2}))}{-\beta(x-a)(x - (b - h \frac{b-a}{2}))} \right] f(x) \\
&\quad - h \frac{b-a}{2} [\alpha(b-x)f(a) + \beta(x-a)f(b)] \\
&\quad + h^2 \frac{(b-a)^2}{8} [\beta(x-a)f'(b) - \alpha(b-x)f'(a)] \\
&\quad + [\alpha(b-x) - \beta(x-a)]F(x) + \beta(x-a)
\end{aligned}$$

Now using (2.4) and (4.2), we get our required result (4.1). \square

Putting $\alpha = \beta = \frac{1}{2}$ in Theorem 5 gives the following result.

Corollary 5. *Let X be a random variable, $F(x)$ cumulative distributive function and f is a probability density function. Then*

$$\begin{aligned}
& \left| \frac{1}{4} \left[(b-x) \left(x - \left(a + h \frac{b-a}{2} \right) \right)^2 - (x-a) \left(x - \left(b - h \frac{b-a}{2} \right) \right)^2 \right] f'(x) \right. \\
& \quad - \frac{1}{2} \left[(b-x) \left(x - \left(a + h \frac{b-a}{2} \right) \right) - (x-a) \left(x - \left(b - h \frac{b-a}{2} \right) \right) \right] f(x) \\
& \quad - h \frac{b-a}{4} [(b-x)f(a) + (x-a)f(b)] \\
& \quad + h^2 \frac{(b-a)^2}{16} [(x-a)f'(b) - (b-x)f'(a)] \\
& \quad \left. + \frac{1}{2} [(b-x) - (x-a)] F(x) + \frac{1}{2} (x-a) \right| \\
& \leq \begin{cases} \left[\begin{aligned} & (b-x) \left\{ \left[x - \left(a + h \frac{b-a}{2} \right) \right]^3 + \left(h \frac{b-a}{2} \right)^3 \right\} \\ & - (x-a) \left\{ \left(h \frac{b-a}{2} \right)^3 + \left[x - \left(b - h \frac{b-a}{2} \right) \right]^3 \right\} \end{aligned} \right] \frac{\|f''\|_\infty}{12}, f'' \in L_\infty[a, b], \\ \\ \left[\begin{aligned} & \frac{1}{(x-a)^q} \left\{ \left[x - \left(a + h \frac{b-a}{2} \right) \right]^{2q+1} - \left(h \frac{a-b}{2} \right)^{2q+1} \right\} \\ & + \frac{1}{(b-x)^q} \left\{ \left(h \frac{a-b}{2} \right)^{2q+1} - \left[x - \left(b - h \frac{b-a}{2} \right) \right]^{2q+1} \right\} \end{aligned} \right]^{\frac{1}{q}} \frac{(b-x)(x-a)}{4(2q+1)^{\frac{1}{q}}} \|f''\|_p \\ , f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \left\{ \begin{aligned} & \frac{1}{2} (b-a) - h(b-a) + \frac{h^2(b-a)}{4} \left[\frac{1}{x-a} + \frac{1}{b-x} \right] \\ & + \left| \frac{1}{2} (a+b-2x) + \frac{h^2(b-a)}{4} \left[\frac{1}{b-x} - \frac{1}{x-a} \right] \right| \end{aligned} \right\} \frac{(b-x)(x-a) \|f''\|_1}{4} \\ , f'' \in L_1[a, b]. \end{cases} \tag{4.3}
\end{aligned}$$

Remark 6. *The above result allow the approximation of $F(x)$ in terms of $f(x)$. The approximation of*

$$R(x) = 1 - F(x)$$

could also be obtained by a simple substitution. $R(x)$ is of importance in reliability theory where $f(x)$ is the probability density function of failure.

Remark 7. *We put $\beta = 0$ in (4.1), assuming that $\alpha \neq 0$ to obtain*

$$\begin{aligned}
& \left| \alpha(b-x) \left\{ \begin{aligned} & \frac{1}{2} \left[\left(x - \left(a + h \frac{b-a}{2} \right) \right)^2 \right] f'(x) \\ & - \left[\left(x - \left(a + h \frac{b-a}{2} \right) \right) \right] f(x) \\ & - h \frac{b-a}{2} f(a) - h^2 \frac{(b-a)^2}{8} [f'(a)] + F(x) \end{aligned} \right\} \right| \\
& \leq \begin{cases} \left[\alpha(b-x) \left\{ \left[x - \left(a + h \frac{b-a}{2} \right) \right]^3 + \left(h \frac{b-a}{2} \right)^3 \right\} \right] \frac{\|f''\|_\infty}{6}, f'' \in L_\infty[a, b], \\ \\ \left[\frac{\alpha^q}{(x-a)^q} \left\{ \left[x - \left(a + h \frac{b-a}{2} \right) \right]^{2q+1} - \left(h \frac{a-b}{2} \right)^{2q+1} \right\} \right]^{\frac{1}{q}} \frac{(b-x)(x-a) \|f''\|_p}{2(2q+1)^{\frac{1}{q}}} \\ , f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \left(\begin{aligned} & \alpha(x-a) - h\alpha(b-a) + \frac{h^2(b-a)}{2} \left(\frac{\alpha}{x-a} \right) \\ & + \left| -\alpha(x-a) + h\alpha(b-a) - \frac{h^2(b-a)}{2} \left(\frac{\alpha}{x-a} \right) \right| \end{aligned} \right) \frac{(b-x)(x-a) \|f''\|_1}{4} \\ , f'' \in L_1[a, b]. \end{cases} \tag{4.4}
\end{aligned}$$

We may replace f by F in any of the equations (4.1),(4.3)and (4.4) so that the bounds are in terms of $\|f''\|_p$, $p \geq 1$. Further we note that

$$\int_a^b F(u) du = uF(u)|_a^b - \int_a^b xf(x) dx = b - E(X).$$

Competing interests:

The authors declare that they have no competing interests.

Authors' contributions:

All authors have contributed equally and significantly in writing this article.

REFERENCES

- [1] P. Cerone, A new Ostrowski Type Inequality Involving Integral Means Over End Intervals, Tamkang Journal Of Mathematics Volume 33, Number 2, 2002.
- [2] P. Cerone and S.S. Dragomir, Trapezoidal type rules from an inequalities point of view, Handbook of Analytic-Computational Methods in Applied Mathematics, CRC Press N.Y. (2000).
- [3] X.L. Cheng, Improvement of some Ostrowski-Grüss type inequalities, Comput. Math. Appl. 42 (2001), 109-114.
- [4] P.L. Čebyšev, Sur les expressions approximatives des integrales definies par les autres prises entre les memes limites, Proc. Math. Soc. Charkov 2 (1882) 93-98.
- [5] S. S. Dragomir and S. Wang, An inequality Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, Computers Math. Applic. 33(1997), 15-22.
- [6] S. S. Dragomir and S. Wang, A new inequality Ostrowski's type in L_1 norm and applications to some special means and some numerical quadrature rules, Tamkang J. of Math. 28(1997), 239-244.
- [7] S. S. Dragomir and S. Wang, A new inequality Ostrowski's type in L_p norm, Indian J. of Math. 40(1998), 299-304.
- [8] S.S. Dragomir and N. S. Barnett, An Ostrowski type inequality for mappings whose second derivatives are bounded and applications, RGMIA Research Report Collection, V.U.T., 1(1999), 67-76.
- [9] G. Grüss, Über das maximum des absoluten Betrages von $\frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{(b-a)^2} \int_a^b f(x) dx \int_a^b g(x) dx$ Math. Z. 39 (1935) 215-226.
- [10] S. Hussain and A. Qayyum, A generalized Ostrowski-Grüss type inequality for bounded differentiable mappings and its applications. Journal of Inequalities and Applications 2013 2013:1.
- [11] W. Liu, Y. Jiang and A. Tuna, A unified generalization of some quadrature rules and error bounds, Appl. Math. Comp. 219 (2013), 4765-4774.
- [12] G.V. Milovanović and J. E. Pecarić, On generalization of the inequality of A. Ostrowski and some related applications, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. (544-576), 155-158, (1976).
- [13] A. Ostrowski, Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, Comment. Math. Hel. 10(1938), 226-227.
- [14] B.G. Pachpatte, On Čebyšev-Grüss type inequalities via Pecarić's extention of the Montgomery identity, J. Inequal. Pure Appl. Math. 7 (1), Art. 108, (2006).
- [15] J.E. Pecarić, On the Čebyšev inequality, Bul. Sti. Tehn. Inst. Politehn "Tralan Vuia" Timişora (Romania) 25 (39) (1) (1980) 5-9.
- [16] A. Qayyum and S. Hussain, A new generalized Ostrowski Grüss type inequality and applications, Applied Mathematics Letters 25 (2012) 1875-1880.
- [17] D. S. Mitrinović, J. E. Pecarić and A. M. Fink, Inequalities for Functions and Their Integrals and Derivatives, Kluwer Academic Publishers, Dordrecht, 1994.

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